Partial Differential Equations

Sonia Reilly

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These notes are based on the first four chapters of Lawrence C. Evans' *Partial Differential Equations.*

1 Introduction

- A *well-posed problem* has a unique solution that depends continuously on the data given in the problem. A classical solution of a k-th order PDE is at least k times continuously differentiable.
- A *weak solution* loosens the regularity requirements.

2 Transport/Advection Equation

- The IVP for transport equation is
$$\begin{cases} u_t + b \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$
for some constant $b \in \mathbb{R}^n$.
- The equation sets a directional derivative in space and time to 0, so u is constant along linear characteristics. (Mathematically, $\frac{d}{ds}(u(x + sb, t + s)) = 0$ reduces to the transport equation.) Tracking each line back to $t = 0$, we get the solution
$$u(x, t) = g(x - tb).$$
This is a classical solution if $g \in C^1$, otherwise a weak solution.
- For the nonhomogeneous problem
$$\begin{cases} u_t + b \cdot \nabla u = f & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$
we've reduced the PDE to an ODE in the parameter s along the characteristic, so
$$\begin{aligned} u(x, t) &= u(x, 0) + \int_{-t}^{0} \frac{d}{ds}(u(x + sb, t + s))ds \\ &= g(x - tb) + \int_{-t}^{0} f(x + sb, t + s)ds \\ &= g(x - tb) + \int_{0}^{t} f(x + (s - t)b, t)ds. \end{aligned}$$

3 Wave Equation

- $u_{tt} - \nabla^2 u = f$
- Derivation: consider a line of masses connected by springs. $F = ma$ using Hooke's law $F(\Delta x) = k\Delta x$ gives the wave equation.

3.1 One dimension

- The solution to
$$\begin{cases} u_{tt} - \nabla^2 u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$
is given by D'Alembert's formula
$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy,$$
which is derived by factoring the wave equation into $(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})u = 0$, solving first a homogeneous transport equation, then an inhomogeneous transport equation, and applying the BCs.
- Can be applied to the positive real line $\mathbb{R}^+$ by applying an odd reflection of the form $\tilde{u}(x,t) = -u(-x,t)$ to u, g, h on $x \le 0$. If u satisfies the PDE on $\mathbb{R}^+$, then the extended $\tilde{u}$ satisfies the PDE on $\mathbb{R}$, so it is given by D'Alembert's formula. Plugging the reflected $\tilde{u}, \tilde{g}, \tilde{h}$ into the formula gives the solution, in which the left-traveling wave reflects off the $x = 0$ boundary.

3.2 In higher dimensions

- In $n \ge 2$ dimensions, we define the average over a sphere
$$U(x; r, t) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)dS(y),$$
and the averages G and H the same way. Then it turns out that this satisfies
$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 & t > 0 \\ U = G,\ U_t = H & t = 0, \end{cases}$$
where $\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r}$ is ∇^2 in spherical coordinates for $U = U(r)$ radially symmetric (proof is messy).
- In 3D, we can choose $\tilde{U} = rU(r)$ and see that
$$\begin{aligned} \tilde{U}_{tt} &= rU_{tt} \\ &= r\left[U_{rr} + \frac{2}{r}U_r\right] \\ &= rU_{rr} + 2U_r = (U + rU_r)_r = \tilde{U}_{rr}, \end{aligned}$$

so $\tilde{U}$ is given by D'Alembert's formula. Then notice $u(x,t) = \lim_{r\to 0^+} \frac{1}{r}\tilde{U}(x;r,t)$, which with D'Alembert's formula gives $u(x,t) = \tilde{G}'(t) + \tilde{H}(t)$. Then plugging in the definitions of $\tilde{G}$ and $\tilde{H}$ and making use of characteristics gives *Kirchhoff's formula*

$$u(x,t) = \frac{1}{\partial B(x,t)} \int_{\partial B(x,t)} [th(y) + g(y) + \nabla g(y) \cdot (y-x)]\, dS(y),$$

which integrates together the contributions of the ICs at all of the points t away.

- To solve in 2D, restrict 3D solution to having a 0 z-component. The calculation is messy. In higher odd dimensions, use more complex substitutions to reduce to 1D wave equation in r. In higher even dimensions, restrict from one dimension higher.
- *Huygens' principle*: Solutions of the wave equations in odd dimensions propagate with a sharp wavefront – they are expressed as integrals over a sphere. Solutions in even dimensions are expressed as integrals over a ball – a disturbance continues to have effects on a point after the wavefront has passed it.
- In 1D these solutions are C^2 as long as $g \in C^2, h \in C^1$. In higher dimensions more smoothness is required for $u \in C^2$. The solutions do not instantly become C^∞, like heat equation solutions.

3.3 Nonhomogeneous problem

- For the nonhomogeneous wave equation IVP,

$$\begin{cases} u_{tt} - \nabla^2 u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, u_t = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

we use Duhamel's principle: define $u(x,t;s)$ by

$$\begin{cases} u_{tt}(\cdot;s) - \nabla^2 u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = 0, u_t(\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{t=s\}. \end{cases}$$

Then $u(x,t) = \int_0^t u(x,t;s)ds$ solves the original IVP. (Verified by plugging in.)

3.4 Uniqueness for the Wave Equation

- There is at most one solution for the nonhomogeneous wave equation on U.
- Proof: Suppose u_1 and u_2 are two solutions, so $w = u_1 - u_2$ satisfies the homogeneous wave equation with 0 ICs and BCs. Define the "energy"

$$E(t) = \frac{1}{2}\int_U w_t^2(x,t) + |\nabla w(x,t)|^2 dx$$

Then

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_U w_t w_{tt} + \nabla w \cdot \nabla w_t dx \\ &= \int_U w_t(w_{tt} - \nabla^2 w)dx = 0, \end{aligned}$$

since there is no boundary term to the integration by parts because the boundary data is 0. Then $E(t) = E(0) = 0$, since $E = \text{const.}$, $w_t(x,0) = 0$, and $w(x,0) = 0 \implies \nabla w(x,0) = 0$. Therefore $w_t, \nabla w = 0$ everywhere, so $w \equiv 0$ by the ICs and BCs, and so $u_1 \equiv u_2$.

4 Fundamental Solutions and Green's Functions

- A fundamental solution $\Phi(x)$ to a PDE with linear operator $\mathcal{L}$ is a function that solves $(\mathcal{L}\Phi)(x) = \delta(x)$.
- Then $u(x) = \int f(x)\Phi(y-x)dx$ solves $(\mathcal{L}u)(x) = f(x)$.
- If we also need to enforce boundary conditions, we use Green's functions. Consider just $\mathcal{L} = -\nabla^2$.
- Green's Theorem says that for any functions u, G, we have

$$\int_\Omega u\nabla^2 G - G\nabla^2 u = \int_{\partial\Omega} (u\nabla G - G\nabla u) \cdot \hat{n}.$$

  It is a simple application of the divergence theorem.
- The idea is that if u satisfies $-\nabla^2 = f$ and $G(x, y)$ satisfies

$$\begin{cases} -\nabla^2 G(x, y) = \delta(y - x) & y \in \Omega \\ G(x, y) = 0 & y \in \partial\Omega, \end{cases}$$

  then Green's theorem reduces to

$$u(x) = \int_\Omega G(x, y)f(y)dV - \int_{\partial\Omega} g(y)(\nabla G(x, y) \cdot \hat{n})dS(y),$$

  which is an explicit expression for u in terms of G.
- Finding a Green's function G is usually hard except in simple geometries. We look for a solution of the form

$$G(x, y) = \Phi(y - x) - v_x(y), \quad x \neq y$$

  such that $v_x(y)$ is a *corrector* function that satisfies

$$\begin{cases} -\nabla^2 v_x(y) = 0 & y \in \Omega \\ v_x(y) = \Phi(y - x) & y \in \partial\Omega. \end{cases}$$

- Note that Green's functions are symmetric: $G(x, y) = G(y, x)$.

5 Laplace's Equation

- Laplace: $\nabla^2 u = 0$, Poisson: $-\nabla^2 u = f$
- Physical interpretation: steady-state diffusion

5.1 Fundamental solution to Poisson's equation

- Assume a radial solution $u(x) = v(r)$, since Laplace's equation is radially symmetric. Recall that in n dimensions,
$$\nabla^2 v(r) = v'' + \frac{n-1}{r} v' = 0.$$
Separation of variables for v' and then integration gives
$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$
for some constants b, c. An integral over $v(r)$ does not solve Laplace's equation everywhere, since $\nabla^2 v$ is not integrable at 0.

- However, this is the form of the fundamental solution of the Poisson equation:
$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, \end{cases}$$
where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ (verifying this is messy). So
$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$
solves Poisson's equation.

5.2 Mean-value formulas

- Mean Value Property: If $u(x)$ is harmonic, it is equal to the average of u over any sphere or ball centered on x:
$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u dS = \frac{1}{|B(x,r)|} \int_{B(x,r)} u dy.$$

- Proof: For the first equality, define
$$\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dS(y) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} u(x + rz) dS(z),$$
where we've removed r from the domain to make it easier to differentiate in r. Then
$$\begin{aligned} \phi'(r) &= \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} \nabla u(x + rz) \cdot z dS(z) \\ &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r} dS(y) \\ &= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \hat{n} dS(y) \\ &= \frac{r}{n} \frac{1}{|B(x,r)|} \int_{B(x,r)} \nabla^2 u(y) dy = 0, \end{aligned}$$

by using the chain rule, the definition of $\hat{n}$, the divergence theorem and the ratio between the surface and volume of a ball, and finally $\nabla^2 u = 0$. Hence $\phi(r) = \text{const.}$, so

$$\phi(r) = \lim_{t \to 0} \phi(t) = u(x)$$

for any r. The second part of the mean value property falls out of expressing an integral over the ball as an integral over surface integrals.

- Converse to Mean Value Property: If $u \in C^2$ satisfies the Mean Value Property for surfaces $\partial B(x,r)$, then u is harmonic.
- Proof: We showed that

$$\phi'(r) = \frac{r}{n} \frac{1}{|B(x,r)|} \int_{B(x,r)} \nabla^2 u(y) dy = 0,$$

but if $\nabla^2 u \not\equiv 0$, we can pick a tiny ball $B(x,r)$ in which the above integral will not be 0, which is a contradiction.

5.3 Maximum principle for harmonic functions

- Maximum Principle: if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic, then the maximum of u in the connected region Ω must be attained on the boundary. Strong Maximum Principle: If it is also attained anywhere on the interior, u is constant. (Both also true of minimum, since $-u$ is also harmonic).
- Proof: If $x_0 \in \Omega$ is an interior point with value $M = \max_{\bar{\Omega}} u$, then we can draw a ball from x_0 to $\partial\Omega$ and the MVP says that the average over the ball is M. Hence $u \equiv M$ in the ball. Thus the set of x with $u(x) = M$ is open, by the definition of openness. It is also (relatively) closed in Ω because it is the inverse image of the closed set $\{M\}$ under a continuous function. Since it is both open and closed, it must equal Ω. The weak maximum principle follows from this.

5.4 Uniqueness of the Poisson Dirichlet BVP

- The Poisson Dirichlet BVP with $f, g \in C^1$

$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

- Proof: If u_1 and u_2 are both solutions, $u_1 - u_2$ solves the homogeneous equation with zero boundary data, which has solution 0 by the maximum (and minimum) principle.

5.5 Smoothness of harmonic functions

- If $u \in C(\Omega)$, and u is harmonic in Ω, then $u \in C^\infty(\Omega)$. (The boundary data need not even be continuous.)
- Proof: do a convolution of u with a mollifier that makes it C^∞, then use the MVP to show that the convolved version is the same as the original u.
- Estimates can be proven on k-th order derivatives of harmonic functions in $\mathbb{R}^n$. A k-th order derivative is bounded by $\frac{C_k}{r^k} \frac{1}{|B(x,r)|} \int_{B(x,r)} u dy$.

5.6 More properties of harmonic functions

- Liouville's Theorem: Any bounded harmonic function on $\mathbb{R}^n$ is constant.
- Proof:
$$u_{x_i}(x) \le \frac{C}{r}\frac{1}{|B(x,r)|}\int_{B(x,r)} u dy \le \frac{C}{R}\max_{y\in B(x,r)} u(y) \to 0$$
as $R \to 0$ since the maximum is bounded, so all derivatives are 0.
- Representation Formula: For $f \in C_c^2(\mathbb{R}^n)$, $n \ge 3$, any solution to Poisson's equation $-\nabla^2 = f$ is given by
$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy + C$$
for some constant C.
- Proof: Since $\Phi(x)$ decays at ∞ for $n \ge 3$, u is a bounded solution. If some $\tilde{u}$ is also a solution, $u - \tilde{u}$ is 0 by Liouville's Theorem.
- If u is harmonic in Ω, it is analytic in Ω. (Proof is messy.)
- Harnack's Inequality: For any $V \subset\subset \Omega$ (i.e., $\bar{V} \subset \Omega$), there is a C_V depending only on V such that
$$\sup_V u \le C_V \inf_V u$$
for all nonnegative harmonic functions u in Ω. (Formalizes the idea that by the MVP, u cannot be very large at one point without being comparably large at all other points.)

5.7 Green's functions for the Poisson equation

- We derived above the expression $G(x,y) = \Phi(y-x) - v_x(y)$ for the Green's function. We seek corrector functions $v_x(y)$ on a few common domains.
- Half-space $x_n \ge 0$: not a bounded region, but it turns out Green's functions still apply. Idea: choose $v_x(y) = \Phi(y - \tilde{x})$, where $\tilde{x}$ is the reflection of x across the boundary of the half-space (i.e., x with the n-th component negative). Now $v_x(y)$ has a singularity outside the region, so it satisfies $\nabla^2 v_x(y) = 0$, and the reflection means that $v_x(y) = \Phi(y-x)$ on the boundary. So the Green's function is
$$G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}).$$
- Unit ball $B(0,1)$: apply same reflection idea but across the sphere, where now $\tilde{x} = \frac{x}{|x|^2}$ is the reflection. Can easily verify that a good Green's function is
$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})).$$
- These functions lead to direct solutions for Poisson's equation on these domains.

5.8 Energy for the Poisson equation

- Dirichlet's principle: Solving the Poisson equation $-\nabla^2 u = f$ is equivalent to minimizing the Dirichlet energy
$$I[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - uf dx.$$
(Note: for Laplace's equation $f = 0$, this is the minimal surface energy.)

6 Heat Equation

- The homogeneous heat equation IVP (Cauchy problem) is

$$\begin{cases} u_t - \nabla^2 u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

6.1 Fundamental solution for the heat equation

- We seek a radial solution of the form

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{|x|}{t^\beta}\right),$$

based on the assumption that $|x|$ should scale with some power of t.

- Plugging into the heat equation, letting $\beta = \frac{1}{2}$ and $\alpha = \frac{n}{2}$ to simplify the equation, and normalizing to integrate to 1, we ultimately get the fundamental solution

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

- Therefore the solution to the heat equation IVP (Cauchy problem) on $\mathbb{R}^n$ is

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy. \end{aligned}$$

- The fundamental solution is the solution to

$$\begin{cases} \Phi_t - \nabla^2 \Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta(x) & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- If $g \geq 0$ and $g \not\equiv 0$, then the solution to the IVP is positive at all points for $t > 0$. So there is infinite propagation speed in a system modeled by the heat equation.

6.2 Inhomogeneous heat equation

- The inhomogeneous IVP is given by

$$\begin{cases} u_t - \nabla^2 u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- Duhamel's principle states that if $u(x,t;s)$ is a solution to the homogeneous IVP with $u(x,t;s) = f(x,s)$ at time $t = s$, then a solution to the inhomogeneous problem with initial data 0 is

$$u(x,t) = \int_0^t u(x,t;s) ds.$$

- The IVP at with initial data $f(x,s)$ at time $t=s$ is given by the representation formula in terms of the fundamental solution, so the solution to the inhomogeneous problem with initial data 0 is
$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dyds.$$

- The solution to the inhomogeneous problem with general initial data g is the sum of this solution with the solution to the homogeneous problem.

6.3 Maximum principles for the heat equation

- For an open bounded region Ω, we define the parabolic cylinder $\Omega_T = \Omega \times (0,T]$. The parabolic boundary $\Gamma_T = \bar{\Omega}_T - \Omega_T$ is the bottom and sides of the cylinder.

- Weak maximum principle: If $u \in C_1^2(\Omega_T) \cap C(\bar{\Omega})$ solves the heat equation in Ω_T, then
$$\max_{\bar{\Omega}_T} u = \max_{\Gamma_T} u,$$
i.e., the maximum in the parabolic cylinder is attained on the parabolic boundary.

- Strong maximum principle: If there exists a point $(x_0, t_0) \in \Omega_T$ such that
$$u(x_0, t_0) = \max_{\bar{\Omega}_T} u,$$
i.e., the maximum is attained away from the boundary, then u is constant in $\bar{\Omega}_{t_0}$. That is, if u attains a maximum (or minimum) at an interior point, then u is constant at all earlier times. (Not true at later times because boundary condition could change.)

- Maximum principle for the Cauchy problem (homogeneous IVP on $\mathbb{R}^n$): Since we don't have a bounded region, we impose a condition on the growth of u,
$$u(x,t) \le Ae^{a|x|^2} \quad 0 \le t \le T$$
for some constants $A, a > 0$. Then
$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g,$$
i.e. the solution never exceeds its maximum initial value.

6.4 Uniqueness of solutions to the heat equation

- The solution to the IBVP
$$\begin{cases} u_t - \nabla^2 u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \end{cases}$$
is unique by the maximum principle applied to the difference between two solutions $u_1 - u_2$.

- The solution to the Cauchy problem is unique by the maximum principle for the Cauchy problem, applied to $u_1 - u_2$. (Note that there are lots of solutions to the Cauchy problem with initial data 0 besides $u \equiv 0$, but these are all nonphysical solutions that blow up very quickly at $|x| \to \infty$.)

6.5 Properties of heat equation solutions

- Regularity: If $u \in C_1^2(\Omega_T)$ solves the heat equation, then $u \in C^\infty(\Omega_T)$.

6.6 Energy methods for the heat equation

- We can also show uniqueness for the IBVP using an energy method. Let $w = u_1 - u_2$ for two solutions u_1, u_2. Then w solves the homogeneous IBVP with 0 ICs and BCs. Let

$$e(t) = \int_\Omega w^2(x,t)dx \quad 0 \le t \le T.$$

Then

$$\begin{aligned}\frac{d}{dt}e(t) &= 2\int_\Omega ww_t dx \\ &= 2\int_\Omega w\nabla^2 w dx \\ &= -2\int_\Omega |\nabla w|^2 dx \le 0,\end{aligned}$$

so $e(t) \le e(0) = 0$ and $w = u_1 - u_2 \equiv 0$.

- Backward uniqueness: If two temperature distributions u_1, u_2 are the same at time T and have the same boundary conditions in $[0,T]$, then $u_1 = u_2$ on all of $[0,T]$. (Proof is another energy method.)

7 Nonlinear Equations

- Consider a general nonlinear first-order PDE:

$$F(\nabla u, u, x) = 0 \quad \text{in } \Omega,$$

subject to $u = g$ on $\Gamma \subseteq \partial\Omega$.

7.1 Method of Characteristics

- Turn PDE into system of ODEs by finding curves in Ω that start at the boundary (giving initial data) and for which u can be solved.

- Parametrize the curve as $x(s)$. Define $z(s) = u(x(s))$ and the vector $p(s) = \nabla u(x(s))$, so $F = F(p, z, x)$.

- The choices of $z(s)$ and $p(s)$ that turn out to solve the PDE are

$$\begin{cases} \frac{d}{ds}p(s) = -\nabla_x F - \frac{\partial}{\partial z}F p(s) \\ \frac{d}{ds}z(s) = \nabla_p F \cdot p(s) \\ \frac{d}{ds}x(s) = \nabla_p F. \end{cases}$$

This gives an ODE in s that can be solved for $x(s)$ and $u(x(s))$.

- In the particular case of a 2D linear/quasilinear system of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u),$$

we can view the equation as a surface in xyu-space (surface plot of u) whose normal is $(a, b, c) \perp (u_x, u_y, -1)$. Then for each point x_0, y_0, u_0 for which we have boundary data we want to construct a curve $x(s), y(s), u(s)$ passing through the point. The characteristic equations are $x'(s) = a$, $y'(s) = b$, $u'(s) = c$.

- Noncharacteristic boundary conditions: There is a unique solution of the IVP on the boundary Γ in a neighborhood of x_0 if Γ is noncharacteristic, i.e.,

$$\nabla_p F(p_0, z_0, x_0) \cdot \hat{n}(x_0) \neq 0,$$

where $\hat{n}(x_0)$ is the outer unit normal of $\partial\Omega$ at x_0. (Characteristics emanating from x_0 cannot be tangent to the boundary/initial data.)

7.2 Conservation Laws

- A conservation law IVP has the form

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

for some given $F(u)$.

- Characteristics of this problem are straight lines along which u is constant.

- The method of characteristics indicates that there is no smooth solution u for all times $t > 0$. Characteristics may cross, and no solution exists after the lines cross. (Example: traffic flow problem where u is car density and $F'(u_0(x))$ is an initial advection velocity/traffic speed. If it is is high for $x < 0$ and low for $x > 0$, cars in $x < 0$ will catch up to the ones in front of them and crash.)

- To admit less smooth solutions, reformulate into weak form by integration by parts. A weak solution satisfies

$$\int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x dxdt + \int_{-\infty}^\infty gvdx|_{t=0} = 0$$

for all test smooth functions v with compact support.

- Jump conditions along shocks: If u is smooth on either side of a smooth curve parametrized by $x(t)$, with u_+ on one side and u_- on the other, the piecewise smooth u is a weak solution if it satisfies the *Rankine-Hugoniot* condition

$$x'(t) = \frac{[F(u)]}{[u]} = \frac{F(u_+(x(t), t)) - F(u_-(x(t), t))}{u_+(x(t), t) - u_-(x(t), t)}.$$

- Example: Burger's equation $u_t + uu_x = 0$. Here $F(u) = \frac{u^2}{2}$. If $u_0(x)$ is monotonically increasing, there is a global solution. Otherwise there will be a shock curve.

8 Other Solution Methods

- Separation of variables: assume $u(x,t) = v(x)w(t)$
- Similarity solutions: look for symmetries, e.g. $u(x) = u(|x|)$
- Transform Methods: Fourier, Radon, Laplace (for $\mathbb{R}^+$)
- Converting nonlinear into linear PDE
- Asymptotics
- Power series

8.1 Fourier transform solutions

- The Fourier transform is defined by

  $$\mathcal{F}[u](k,t) = \int_{\mathbb{R}} u(x,t)e^{-ikx}dx$$

  and the inverse transform is defined by

  $$\mathcal{F}^{-1}[\hat{u}](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(x,t)e^{ikx}dx.$$

- For some $u(x,t)$, $\mathcal{F}[\frac{\partial u}{\partial x}] = ik\mathcal{F}[u](k,t)$.
- Can solve a PDE (e.g., wave equation, heat equation) by taking a Fourier transform in x of the equation, solving the resulting ODE in time, and taking an inverse Fourier transform.

8.2 Power series

- *Cauchy-Kovalevskaya Theorem*: A PDE given by an analytic function of x, u, and derivatives of u with boundary data on a noncharacteristic surface has a local solution given by a power series near the boundary data.
- Recall analytic means that a power series converges locally to the function (stronger condition than C^∞).
- An example of a noncharacteristic surface is $t = 0$.