Fluid Dynamics

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These notes are based on Antoine Cerfon's Fall 2021 Fluid Dynamics lecture notes.

1 Describing fluids

- Newtonian fluid: $\tau = \mu \frac{du}{dy}$, i.e., shear stress proportional to rate of deformation. Non-Newtonian if μ is not constant. Elastic solid if stress is linearly related to strain, not deformation rate.
- Lagrangian view: describe fluid flow by trajectories of fluid elements (path lines). Eulerian view: describe flow by velocity at each point over time.
- Stream lines: tangent to flow field at every point at a given time. For flow $\vec{u} = \langle u, v, w \rangle$, they are computed by solving $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.
- In a steady flow, $\frac{d\vec{u}}{dt} = \vec{0}$, stream lines, path lines and streak lines (all fluid elements that have past through a point) coincide.
- Material derivative: change in property θ of the fluid

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \vec{u} \cdot \nabla \theta.$$

  A *material property* is a property with material derivative 0.
- Incompressibility (volume conservation) $\iff \nabla \cdot \vec{u} = 0$
- Convection theorem (change in the total quantity $f(\vec{x}, t)$ in a region Ω_t whose boundaries move with the fluid):

$$\frac{d}{dt}\left(\int_{\Omega_t} f(\vec{x}, t) d\vec{x}\right) = \int_{\Omega_t} \frac{\partial f}{\partial t} d\vec{x} + \int_{\partial \Omega_t} f \vec{u} \cdot \vec{n} dA$$

2 Conservation of mass and momentum

- Conservation of mass:

$$\begin{aligned} \frac{DM}{Dt} = \frac{D(\rho dV)}{Dt} &= \frac{D\rho}{Dt} dV + \rho \frac{DdV}{Dt} = 0 \\ &\iff \left(\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho\right) dV + \rho(\nabla \cdot \vec{u}) dV = 0 \\ &\iff \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0. \end{aligned}$$

- Therefore fluid flow is incompressible if there are no mass sources and $\rho \equiv$ const. (Example application: 1D flow in a pipe of varying diameter – velocity is inversely proportional to cross-sectional area.)
- For an incompressible fluid without constant uniform ρ, conservation of mass reduces to $\frac{D\rho}{Dt} = 0$.
- A fluid element experiences body forces (e.g., gravity) and surface forces (normal and shear stresses).
- The surface forces are expressed in terms of a stress tensor Π with entries Π_{ij} that represent the force per unit area in direction j on the surface with normal vector in direction i. Then $\vec{F}_{\text{surface}} = \vec{n} \cdot \Pi$.
- The force per unit volume due to surface forces is $\vec{F}_s = \nabla \cdot \Pi$ (from divergence theorem on a surface integral of $\vec{F}_{\text{surface}}$).
- Conservation of momentum:
$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \rho \vec{g} + \nabla \cdot \Pi.$$

3 Ideal fluids

- Ideal fluid: no viscosity and incompressible, i.e., the only surface force is pressure, $\Pi = -pI$.
- **Euler equations**: conservation of mass and momentum in an ideal fluid
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$
$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \rho \vec{g} - \nabla p.$$
These are 4 equations for 5 unknowns – we need one more, e.g., incompressibility.
- Hydrostatics: $\vec{u} = 0$
  - Then $\nabla p = \rho \vec{g}$, so $\frac{dp}{dx} = \frac{dp}{dy} = 0$, and $\frac{dp}{dz} = -\rho g$. If ρ is uniform, $p = p_0 - \rho g z$.
  - Pascal's Law: all points at the same depth are at the same pressure.
  - Archimedes' Principle: the buoyancy force, i.e., the net force of fluid pressure on a suspended object, is equal to the weight of the fluid displaced.
- Ideal fluids with $\rho =$ const.
  - Conservation of mass implies $\nabla \cdot \vec{u} = 0$, which provides the last equation needed for a unique solution to the Euler equations.
  - In steady flow $\frac{\partial \vec{u}}{\partial t} = 0$, *Bernoulli's Theorem* follows from the Euler equations: The Bernoulli function $H = \frac{u^2}{2} + gz + \frac{p}{\rho}$ is constant along streamlines, i.e., $\vec{u} \cdot \nabla H = 0$. (Applications: fluid speeding up in a narrowing pipe, Pitot tube.)
- For Euler momentum equation, we need one IC and one BC.
- At a fixed rigid wall, $\vec{u} \cdot \vec{n} = 0$ (implies that an element at the wall only moves along the wall).
- At a moving wall or fluid interface, $(\vec{u} - \vec{u}_S) \cdot \vec{n} = 0$, i.e., the normal component of the velocity is continuous while the tangential component may be discontinuous.

4 Vorticity

- The Jacobian $\nabla\vec{u}$ can be decomposed into symmetric and antisymmetric parts: $\frac{\partial u_i}{\partial x_j} = S_{ij} + \frac{1}{2}R_{ij}$, where $S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j})$ is the *strain rate tensor* and $R_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_j}$ is the *rotation tensor*. The diagonal elements of S_{ij} are the linear deformations (elongation and contraction), and the off-diagonal elements are shear deformations.

- The *vorticity* of a fluid flow is defined as

$$\vec{\omega} = \nabla \times \vec{u}.$$

- Vorticity is local: shear flow has no global rotation, but has nonzero vorticity, while line vortex flow ($\vec{u} = \frac{1}{r}\vec{e}_\theta$) has global rotation but zero vorticity (except at 0).

- Circulation around a closed curve C:

$$\Gamma = \oint_C \vec{u} \cdot d\vec{r} = \iint_S \vec{\omega} \cdot \vec{n}\, dA,$$

  i.e., the flow along a closed curve or the flux of vorticity across any surface bounded by the curve.

- For irrotational/potential flow, there can still be nonzero circulation around an object (e.g., around the origin of a line vortex).

- *Kelvin's circulation theorem*: For an ideal fluid subject to a volume force of the form $\rho\nabla\chi$, the circulation around a material curve (a curve that is advected with the fluid) is constant in time if $\rho =$ const. or the fluid is barotropic ($p = p(\rho)$).
  - As a result, the flux of vorticity of an ideal barotropic fluid through a surface is an invariant of the flow.
  - Implies that there are three ways to create or destroy vorticity:
    * volume forces that cannot be written as the gradient of a potential (e.g., Coriolis force)
    * non-barotropic fluid (e.g., vertical pressure gradient due to gravity, horizontal density gradient causing baroclinic instability in which denser fluid rotates to the bottom)
    * viscosity (e.g., viscosity at a wall causing shear flow)

- *Vortex lines* follow the vorticity field like stream lines follow the velocity field: $\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$. The vortex lines passing through any closed curve form a *vortex tube*, and the strength of the vortex tube is the circulation around any of the closed curves along it.

- Helmholtz's Laws (for ideal barotropic or constant density fluid – all consequences of Kelvin's circulation theorem):
  - Vortex lines are material lines (move with the fluid)
  - A vortex tube cannot end within the fluid: it ends at the boundary or forms a closed loop
  - Fluid elements free of vorticity remain free of vorticity

- The strength of a vortex tube moving with the fluid is invariant in time

- Vortex stretching: if $\rho = \text{const.}$, the vorticity increases as a vortex tube is stretched (by conservation of mass in the tube and invariance of tube strength).

- **Vorticity equation** (for constant, uniform density and conservative volume forces):

$$\begin{aligned} &\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \nabla \chi \\ \iff &\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{u^2}{2} \right) - \vec{u} \times \vec{\omega} = -\frac{1}{\rho} \nabla p + \nabla \chi \end{aligned}$$

Taking the curl and applying $\nabla \times (\vec{u} \times \vec{\omega}) = \cancelto{0}{\vec{u} \nabla \cdot \vec{\omega}} - \cancelto{0}{\vec{\omega} \nabla \cdot \vec{u}} + \vec{\omega} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{\omega}$, we have

$$\begin{aligned} &\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} - \vec{\omega} \cdot \nabla \vec{u} \\ \iff &\boxed{\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla \vec{u}.} \end{aligned}$$

- The rhs of the vorticity equation is called the vortex stretching term: if $\vec{\omega} = \omega \hat{z}$, the equation reduces to $\frac{D\omega}{Dt} = \omega \frac{\partial u_z}{\partial z}$, so if $\frac{\partial u_z}{\partial z} \geq 0$, ω increases in time as the fluid elements are stretched.

- In 2D, the equation reduces to $\frac{D\omega}{Dt} = 0$, due to the absence of vortex stretching.

- In an ideal barotropic fluid, the vorticity equation applies to $\frac{\vec{\omega}}{\rho}$ instead of $\vec{\omega}$.

- The velocity field induced by a vorticity field $\vec{\omega}$ is given uniquely by

$$\vec{u}(\vec{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{(\vec{y} - \vec{x}) \times \vec{\omega}(\vec{y})}{||\vec{x} - \vec{y}||^3} d\vec{y}.$$

  as long as $\vec{\omega}$ is smooth and decays faster than $\frac{1}{r^2}$ at ∞.

- Vortex interaction:
  - Consider point vortices in 2D of strength Γ, so the induced $\vec{u}$ is $\langle u_r, u_\theta \rangle = \langle 0, \frac{\Gamma}{2\pi r} \rangle$.
  - Two vortices: Each will move with the velocity induced by the other, so with opposite spins they will move side-by-side along parallel lines, and with the same spins they will circle each other.
  - Vortex near a wall: To enforce $\vec{u} \cdot \vec{n} = 0$ at the wall, we use the method of images by noting that this is equivalent to having a mirrored vortex on the other side of the wall, so the initial vortex will travel up the wall.

5 Potential flow

- If $\nabla \times \vec{u} = 0$ in a simply connected domain, then $\vec{u} = \nabla \phi$, i.e., irrotational flow is potential flow. The potential ϕ is defined up to a function of t. (In a domain that is not simply connected, the ϕ may not be single-valued, e.g., $\phi(r, \theta) = \frac{\Gamma \theta}{2\pi}$ for a point vortex.)

- Incompressible potential flow in 2D: Given $\vec{u} = \nabla\phi$ and $\nabla \cdot \vec{u} = 0$, we have $\nabla^2\phi = 0$. The fundamental solution is $\phi(r,\theta) = \frac{Q}{2\pi}\ln r \implies \vec{u}(r,\theta) = \frac{Q}{2\pi r}\vec{e}_r$, i.e., point source flow. Another solution is uniform flow, so a linear combination of the two is also a solution.

- For 2D incompressible potential flow, we can define a *streamfunction* $\psi(\vec{r}) = \int_\gamma = \vec{u} \cdot \vec{n}\, dS$ which is equal for any curve γ from some arbitrary $\vec{r}_0$ to $\vec{r}$. The streamfunction is constant along streamlines: $\vec{u} \cdot \nabla\psi = 0 \iff \frac{\partial\psi}{\partial x} = -v,\ \frac{\partial\psi}{\partial y} = u \iff \vec{u} = \nabla\psi \times \vec{e}_z$.

- The streamlines are everywhere orthogonal to the countour lines of the velocity potential ϕ.

- The streamfunction also satisfies Laplace's equation, because

$$\nabla \times \vec{u} = \nabla \times (\nabla\psi \times \vec{e}_z) = -\vec{e}_z \nabla^2\psi = \vec{0} \implies \nabla^2\psi = 0.$$

- Boundary conditions for $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$
  - At infinity: for uniform flow $\langle U, V\rangle$ that is diverted around a body, we have $\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\rangle = \langle U, V\rangle$ and $\langle \frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\rangle = \langle U, V\rangle$ at infinity.
  - At a solid boundary (ideal fluid): $\vec{u} \cdot \vec{n} = 0 \implies \nabla\phi \cdot \vec{n} = 0$ and $\nabla\psi \cdot \vec{t} = 0$ (for tangent vector $\vec{t}$).

- We can express ψ and ϕ as components of a complex potential

$$w(z) = \phi(x, y) + i\psi(x, y)$$

  because $\vec{u} = \nabla\phi = \nabla\psi \times \vec{e}_z \implies \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$ and $\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$, which are the Cauchy-Riemann equations.

- Since $\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = u_x - iu_y$, we have $|\vec{u}| = \left|\frac{dw}{dz}\right|$.

- *Milne-Thomson's circle theorem*: If $f(z)$ is a complex potential whose singularities lie in $|z| > a$, then $w(z) = f(z) + \overline{f(a^2/z)}$ is the complex potential with the same singularities in $|z| > a$, and $|z| = a$ as a streamline.
  - Used for flow around a cylinder. For uniform flow $\vec{u} = \langle U, 0\rangle$ we have $f(z) = Uz$, so $w(z) = U(z + R^2/z)$ is the complex potential of the flow around the cylinder because it has a streamline along the boundary $|z| = R$.

6 Lift and drag in ideal fluids

- Conformal mapping (for 2D potential flow): If $w(z)$ is a complex potential in one domain, and f is a conformal map (analytic function with derivative nowhere 0) from that domain to another, then $W(Z) = w(f^{-1}(Z))$ is a complex potential in the second domain.

- Equipotential lines and stream lines in the original domain are mapped to equipotential lines and stream lines in the new domain.

- Key idea is to compute flow with simple boundary conditions and use conformal mapping and the Riemann mapping theorem to map the flow to a more complex domain.

- Example: Joukowski transformation $f(z) = z + \frac{b^2}{z}$ maps circles to ellipses

- The map gives uniform flow at infinity when $\frac{dW}{dZ} = \frac{dw}{dz}\frac{d(f^{-1})}{dz} = \frac{u_x - iu_y}{f'(z)}$ is defined in the limit $z \to \infty$.

- General solution for flow past a circular cylinder: potential $w(z) = U\left(z + \frac{R^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z$, where Γ is the circulation around the cylinder (circulation-free potential plus point vortex potential).

- For $|\Gamma| < 4\pi RU$, there are two stagnation points, on the front and back of the cylinder. For $|\Gamma| > 4\pi RU$, there is one stagnation point away from the cylinder surface (from solving for velocity and setting to 0).

- The fluid applies only a lift force, not a force in the direction of the fluid. The lift force is $L = -\rho U\Gamma$. (From applying Bernoulli's theorem to the streamline on the cylinder surface, plugging in u_θ, and observing that $p(\pi - \theta) = p(\theta)$, so there is no pressure differential between the front and back of the cylinder. Then integrate the vertical force $-pR\sin\theta\,\hat{y}\,d\theta$ over the circle to get $-\rho U\Gamma\hat{y}$.)

- *Theorem of Blasius*: An object with contour C in a steady flow with potential $w(z)$ experiences a force $\hat{F}$ such that
$$F_x - iF_y = \frac{1}{2}i\rho\oint_C\left(\frac{dw}{dz}\right)^2 dz.$$
Gives another way to derive lift. (Proof: $d\hat{F} = -p\,ds\,\hat{n}$, C as streamline, Bernoulli theorem, $||\hat{u}|| = \frac{dw}{dz}e^{i\theta}$ at C, $dz = e^{i\theta}ds$, etc.)

- *Kutta-Joukowski Lift Theorem*: A 2D object with contour C in a flow that is $\hat{u} = \langle U\cos\alpha, U\sin\alpha\rangle$ at infinity and has circulation Γ around the object experiences a force
$$\hat{F} = \rho U\Gamma\langle\sin\alpha, -\cos\alpha\rangle.$$
(Proof: follows from Theorem of Blasius using a contour integral around a circle around the object.)

- Airfoil theory for a line segment
  - Joukowski map $Z = z + \frac{b^2}{z}$ maps circle with radius $R = b$ to line segment $[-2b, 2b]$. Can use this to derive $W(Z)$ and $\frac{dW}{dZ}$.
  - The flow has singularities at the leading and trailing edges, which are resolved by considering viscous flow. In practice, the trailing edge flow is smooth, which happens when the *Kutta-Joukowski condition* $\Gamma = -4\pi Ub\sin\alpha$ is satisfied. Experimentally, this is true for $|\alpha| \ll 1$.
  - In rapid, unsteady flow, the airfoil experiences vortex shedding from the trailing edge. The vortices compensate for the circulation around the airfoil in Kelvin's theorem applied to a large contour around the problem. They carry energy downstream and are associated with drag on the wing.
  - The singularity at the leading edge requires considering viscous flow. In practice an eddy forms at the leading edge and the streamline separates from the boundary, decreasing lift and increasing drag (stalling). Using a rounded leading edge in a real airplane wing avoids this. The Joukowski map on some circles centered away from the origin results in more realistic airfoil contours.

- Ideal flow in 3D
  - For axisymmetric (radially symmetric) incompressible flow $\hat{u} = \langle u_r(r,z), u_\phi(r,z), u_z(r,z)\rangle$, we can define a *Stokes streamfunction* Ψ such that $\hat{u} = \nabla\Psi \times \nabla\Phi + u_\phi \hat{e}_\phi$.
  - Butler's Theorem is a direct analogue to Milne-Thomson in 3D for axisymmetric incompressible flow with $u_\phi = 0$. Gives flow around sphere.
  - D'Alembert's paradox: as in 2D, a 3D steady, uniform flow of an ideal fluid past a fixed body gives no drag on the body. Viscous effects are required to explain the drag we see in reality.

7 Waves in an ideal and incompressible fluid

- Surface wave:
$$z = \eta(x,t) = A\cos\left[\frac{2\pi}{\lambda}(x-ct)\right] = A\cos(kx - \omega t),$$
with amplitude A, wavelength λ, phase velocity c, period $T = \frac{\lambda}{c}$, frequency $\omega = \frac{2\pi}{T}$, and wavenumber $k = \frac{2\pi}{\lambda}$.
- If the fluid is moving with bulk velocity U, the observed frequency is $\omega_0 = \omega + kU$ (Doppler shift).
- Assume 2D (xz-plane), irrotational (good assumption if starts at rest and has only conservative forces, by vorticity equation), incompressible. Hence potential flow: $\nabla^2\phi = 0$.
- Boundary conditions: At lower boundary, $\frac{\partial\phi}{\partial z}|_{z=-H} = 0$. At upper boundary, we take $F(x,z,t) = z - \eta(x,t) = 0$, so since the boundary is a material curve,
$$\frac{\partial F}{\partial t} + \vec{u}\cdot\nabla F = 0 \iff \frac{\partial\eta}{\partial t} + u_x\frac{\partial\eta}{\partial x} = u_z \quad \text{at } z = \eta(x,t).$$
Additionally,
$$\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + g\eta(x,t) = 0 \quad \text{at } z = \eta(x,t),$$
from Bernoulli equation, since ϕ is defined up to a constant and $p = p_{atm}$ at surface if no surface tension. (Second BC necessary for dispersion relation.)
- Solving Laplace's equations with these BCs is hard, so linearize the surface BCs assuming $A \gg \lambda$:
$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z}(x,0,t), \quad \frac{\partial\phi}{\partial t}(x,0,t) = -g\eta(x,t)$$
Velocity solution is circular in the x-direction but has cosh/sinh dependence in the z-direction.
- Surface waves in infinitely deep fluid, $H \to \infty$:
  - The fluid moves in circles that are exponentially smaller away from the surface.
$$\langle u_x, u_z\rangle = A\omega e^{kz}\langle\cos(kx-\omega t), \sin(kx-\omega t)\rangle.$$
  - Plugging ϕ into the Bernoulli BC gives the dispersion relation
$$\omega^2 = gk \implies c^2 = \frac{\omega^2}{k^2} = \frac{g}{k}.$$

- Group velocity of a wave packet centered around wave number k is $c_g(k) = \frac{d\omega}{dk}$, so for deep water waves, $c_g(k) = \frac{1}{2}\sqrt{\frac{g}{k}}$. So long wavelengths travel faster. The phase velocity (velocity of peaks and troughts) is twice the group velocity.

- Shallow water surface waves:
  - For finite depth, $\omega^2 = gk\tanh(kH)$, so for shallow water, $\omega^2 = gHk^2$. Hence $c^2 = \frac{\omega^2}{k^2} = gH = \text{const.}$, so all waves travel at the same speed $c_g = \sqrt{gH}$.
  - The particle paths are ellipses that get flatter with depth.

- Capillary waves:
  - Surface tension causes a pressure jump $p_{atm} - p = \sigma\frac{\partial^2\eta}{\partial x^2}$, which can be plugged into Bernoulli to get dispersion relation $\omega^2 = gk(1 + \frac{\sigma k^2}{\rho g})$. Therefore surface tension matters for large k, i.e., short wavelengths.
  - In the limit $\frac{\sigma k^2}{\rho g} \gg 1$, we get *capillary waves* with $\omega^2 = \frac{\sigma k^3}{\rho}$. In this case shortest wavelengths travel the fastest, phase velocity is slower than group velocity (peaks appear to move backwards).

8 Waves in compressible inviscid fluids

- If $\nabla \cdot \vec{u} \neq 0$, we need another equation to have a unique solution to the Euler equations.

- Example: in hydrostatic case, momentum equation reduces to $\frac{dp}{dz} = -\rho g$. To solve, we apply ideal gas law $p = \rho RT$ to get $p(z) = p_0 e^{-gz/RT}$ (e.g., approximation for pressure in atmosphere).

- Conservation of energy equation (for ideal fluid):

$$\rho\frac{De}{Dt} = -p\nabla \cdot \vec{u} - \nabla \cdot \vec{q},$$

  i.e., the change in the internal (microscopic) energy is the work minus the outflow of heat. To close Euler's equations need model for heat conduction, e.g. Fourier's law $\vec{q} = -k\nabla T$ (heat conduction k), and equations of state $T(\rho, p)$ and $e(\rho, p)$ from thermodynamics.

- For an ideal gas, $e \propto T$, and $T \propto \frac{p}{\rho}$. The energy equation can then be nondimensionalized using the Mach number $Ma^2 = \frac{\rho_0 v_0^2}{\gamma p_0}$.

- If $Ma \ll 1$, the energy equation reduces to incompressibility. If $Ma \sim 1$, it reduces to $\frac{D}{Dt}(\frac{p}{\rho^\gamma}) = 0$. In the adiabatic case (gas dynamics happen fast enough that heat flow is negligible), this gives the compression equation $\frac{p}{p_0} = (\frac{\rho}{\rho_0})^\gamma$.

- Sound Waves
  - Linearize mass conservation, momentum conservation, and energy equation around $\vec{u} = 0$ and ρ_0, p_0 constant to get that p satisfies the wave equation.
  - Sound speed is $c_s^2 = \frac{\gamma p_0}{\rho_0}$. Sound waves travel with the sound speed of the fluid and are not dispersive.

9 The Navier-Stokes equations

- In a nonideal fluid, there are surface forces that are not normal to the surface:

$$\Pi_{ij} = -p\delta_{ij} + \tau_{ij},$$

where $\tau_i j$ is a symmetric matrix called the *deviatoric stress tensor.*

- In a Newtonian fluid, τ_{ij} is linear in $\nabla\vec{u}$. Assuming the fluid is also isotropic,

$$\tau_{ij} = \lambda S_{kk}\delta_{ij} + 2\mu S_{ij}$$

where S_{ij} is the *strain rate tensor*, and $S_{kk} = \nabla\cdot\vec{u}$. Setting $\mu_v = \lambda + \frac{2}{3}\mu$ the bulk viscosity, we get

$$\Pi_{ij} = -p\delta_{ij} + 2\mu\left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right) + \mu_v S_{kk}\delta_{ij}.$$

The Stokes assumption is μ_v, which applies whenever a fluid is not being rapidly compressed (e.g., sound waves, shock waves.) If the fluid is incompressible, $S_{kk} = \nabla\cdot\vec{u} = 0$.

- From plugging this into the momentum equation with general $\nabla\cdot\Pi$, we get compressible Navier-Stokes

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u} + \left(\mu_v + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\vec{u})$$

and incompressible Navier Stokes

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u}.$$

- Boundary Conditions
  - Need two, because now we have 2nd order derivatives in space
  - Normal BC: $\vec{n}\cdot(\vec{u} - \vec{u}_S) = 0$
  - Tangential BC: either no-slip $\vec{u} = \vec{u}_S$
  - Or zero tangential stresses, e.g. for shear flow $u_x(y)$, $\frac{du_x}{dy} = 0$.

- Nondimensionalized Navier-Stokes: scale by characteristic L, T, U to get

$$St\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \frac{1}{Fr}\vec{e}_z + \frac{1}{Re}\nabla^2\vec{u},$$

where the Strouhal number is $St = \frac{L}{TU}$, the Froude number is $Fr = \frac{U^2}{gL}$, and the Reynolds number is $Re = \frac{UL}{\nu}$ for $\nu = \frac{\mu}{\rho}$ the kinematic viscosity.

- High Reynolds number: like ideal fluid, except at a boundary layer with thickness $\propto \frac{1}{\sqrt{Re}}$. Steady flow at high Reynolds number is unstable to small disturbances, and may become turbulent.

- Low Reynolds number: Well ordered, laminar, reversible. Turbulence is quickly damped.

- Vorticity equation for viscous flow:

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\nabla\vec{u} + \nu\nabla^2\vec{\omega},$$

because $\nabla\times(\nabla^2\vec{u}) = \nabla\times(\nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u})) = -\nabla\times(\nabla\times\vec{\omega}) = \nabla^2\vec{\omega}$.

10 Laminar flow

- Special cases of Navier-Stokes where the nonlinear advective term is 0
- Fully developed flow in a pipe, $\vec{u} = u_x(y)\vec{e}_x$.
- Steady flow between parallel plates
  - Lower plate is fixed, upper plate moves with velocity $U\vec{e}_y$.
  - Given $\vec{u} = \langle u_x(y), u_y(y)\rangle$, incompressibility $\implies \frac{du_y}{dy} = 0 \implies u_y = \text{const.}$, so by BCs $u_y = 0$.
  - Hence $\vec{u} \cdot \nabla\vec{u} = u_x \frac{\partial u_x}{\partial x}\vec{e}_x = \vec{0}$.
  - Then Navier-Stokes x-component is $\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \implies p(x, y) = p(x)$, and the y-component is
  $$\frac{1}{\rho}\frac{dp}{dx} = \nu\frac{d^2u_x}{dy^2},$$
  which implies that both terms are constant, since the LHS depends only on x and the RHS depends only on y. Solving gives
  $$u_x(y) = \frac{U}{h}y - \frac{1}{2\mu}\frac{dp}{dx}y(h - y).$$
  - If there is no pressure gradient in the direction of the flow, $\frac{dp}{dx} = 0$, then the flow profile is linear – called *plane Couette flow*. Shear stress $\tau = \mu\frac{du_x}{dy} = \mu\frac{U}{h}$ is uniform.
  - If the upper plate does not move, $U = 0$, then the flow is entirely driven by the pressure gradient and the flow profile is parabolic – *plane Poiseuille flow*.
- Steady flow in cylindrical geometry
  - Circular Poiseuille flow occurs in flow through a pipe
  - Circular Couette flow occurs between two concentric cylinders rotating in opposite directions. Solution is a superposition of rigid body rotation ($u(r) \propto r$) and a line vortex ($u(r) \propto \frac{1}{r}$).

11 Stokes flow

- Stokes flow is a type of laminar flow where we let $Re \ll 1$ but keep $|Re\nabla p| \sim 1$, so incompressible Navier-Stokes reduces to
$$\nabla^2\vec{u} = \nabla p, \quad \nabla \cdot \vec{u} = 0.$$
- Elliptic PDE with no time derivative: $\vec{u}$ responds instantaneously to the BCs.
- Time reversibility of Stokes flow: since the solution is uniquely determined by the velocity of the boundary given no-slip BCs, the velocity is exactly reversed if the boundary velocity is reversed (corn syrup dye experiment).
- Stokes flow around a sphere: messy calculations, but solving Stokes equations in spherical coordinates is straightforward. To compute net drag, integrate $p\hat{n}$ and the viscous stresses over the surface of the sphere to get $D = 6\pi\mu U a\vec{e}_z$.

- Stokes flow around an infinite cylinder: The Stokes equations have no solution in this case when we use a BC at the cylinder and one at infinity (Stokes' paradox). The reason is that the Stokes approximation does not apply outside of a region near the cylinder.