# Spring 2026: Mathematical Statistics
## Recitation Worksheet 4
Feb. 20, 2026

1. Suppose instead of a large political poll, like in the last recitation, now we conduct a poll of 36 out of 71 people (sampling without replacement). 12 of those polled are in favor of the measure. Find a 95% confidence interval for the proportion that is in favor. (Challenge: for problems like these where $X_i \sim \mathrm{Ber}(p)$, find a general expression for the CI of $p$ in terms of $\hat{p} = \bar{X}_n$.)

2. [7.36] With simple random sampling, is $\bar{X}^2$ an unbiased estimate for $\mu^2$? (Note that the textbook uses "simple random sampling" to mean sampling without replacement.) If not, what is the bias?

3. [7.38] Is $\frac{1}{n}\sum_{i=1}^{n} X_i^3$ an unbiased estimator for $\frac{1}{N}\sum_{i=1}^{N} x_i^3$, if we sample with replacement? What if we sample without replacement?

**Midterm Problems**

1. **True/False.** Determine whether each statement is always true. No justification is needed. (1 pt each) (5 points)

   (a) If $A \subseteq B$ and $\mathbb{P}(B) > 0$, then $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$.

   (b) If $X$ and $Y$ are independent, then $\mathrm{Cov}(X, Y) = 0$.

   (c) If $\mathrm{Cov}(X, Y) = 0$, then $X$ and $Y$ are independent.

   (d) For any random variable $X$ with finite variance, $\mathrm{Var}(2X + 3) = 4\mathrm{Var}(X)$.

   (e) The MGF of a sum of independent random variables equals the product of their MGFs.

2. Let $U_1$ and $U_2$ be independent $\mathrm{Unif}[0, 1]$ random variables. Define

$$X = \max(U_1, U_2).$$

   (a) Find the CDF and PDF of $X$. (3 points)

   (b) Compute $\mathbb{E}[X]$. (2 points)

3. Let $X$ and $Y$ be independent Exponential($\lambda$) random variables, with density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the PDF of $X + Y$. (5 points)

4. Let $X, Y \sim \mathrm{Bern}(\frac{1}{2})$ be two independent outcomes of fair coin tosses, and

$$Z = \begin{cases} X & \text{if } X = 0 \\ Y & \text{if } X = 1. \end{cases}$$

   (a) Find the marginal distribution of $Z$. (2 points)

   (b) Compute $\mathbb{E}[X|Z]$. (3 points)

5. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Exponential}(\lambda)$ so $\mathbb{E}[X] = 1/\lambda$ and $\mathrm{Var}(X) = 1/\lambda^2$.

   (a) Show that $\bar{X}_n$ converges in probability to $1/\lambda$. (2 points)

   (b) Use the CLT to find an interval $[a, b]$ (which may depend on $\bar{X}_n$ and $n$) such that $\mathbb{P}(\lambda \in [a, b]) \approx 0.95$. You may assume that $n \geq 5$.
   *Hint:* $\mathbb{P}(|Z| \geq 2) \approx 0.95$ for $Z \sim N(0, 1)$. (3 points)

**Bonus:** Suppose $n$ people each independently and uniformly choose a number from $\{1, \ldots, n\}$. What is the expected number of numbers that nobody chose? (Extra 2 points)