Mathematical Statistics Recitation 2

Section 002 (Prof. Niles-Weed)

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February 6, 2026

Reminder: Midterm 1 on Thursday Feb 12! Covers Lectures 1-6 (everything up through today).

Lecture Review

Lecture 5

Covariance

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$
$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$$

Correlation coefficient:

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

- $-1 \le \rho \le 1$
- If X, Y are independent, then $\rho = 0$ (not the converse).
- $\rho = \pm 1$ indicates perfect (linear) correlation.

Conditional Expectation and Variance

Conditional Expectation:

$$\mathbb{E}[Y \mid X = x] = \int y\, f_{Y|X}(y \mid x)\, dy,$$

which is a function of x. $\mathbb{E}[Y \mid X]$ is a random variable.

Law of Total Expectation:

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]]$$

(inner expectation in Y, outer expectation in X)

Law of Total Variance

$$\mathrm{Var}(Y) = \mathrm{Var}(\mathbb{E}[Y \mid X]) + \mathbb{E}[\mathrm{Var}(Y \mid X)]$$

Lecture 6

Convergence of Random Variables
Convergence in probability:

$$X_n \xrightarrow{p} X \quad \text{if} \quad \mathbb{P}(|X_n - X| > \varepsilon) \to 0 \quad \forall \varepsilon$$

Convergence in distribution:

$$X_n \xrightarrow{d} X \quad \text{if} \quad \lim_{n\to\infty} F_n(x) = F(x),$$

where F_n, F are CDFs of X_n, X.
Moment Generating Functions

$$M(t) = \mathbb{E}(e^{tX})$$

uniquely determines the distribution if it exists.
Moment generating because k-th derivative at 0 is k-th moment:

$$M^{(k)}(0) = \mathbb{E}(X^k)$$

$$M_n(t) \to M(t) \quad \text{in open interval around 0} \implies F_n(t) \to F(t)$$

(used to prove CLT).
Law of Large Numbers (LLN)

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}[X_i] \quad \text{as } n \to \infty$$

if the X_i are iid. (Proof: Uses Chebyshev's inequality.)
Central Limit Theorem:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{as } n \to \infty$$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

for X_i i.i.d. with mean μ, variance σ^2, and MGF defined in neighborhood of 0.
Can interpret as

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{approximately, for large } n$$

(rule of thumb: $n \geq 30$)

Reminder from probability: Gamma Distribution

pdf with parameters α, λ:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

In particular, $\Gamma(n) = (n-1)!$ for integers.
Note: only the constant scaling is complicated. At heart, the distribution is just a mix of powers of x and exponentials of x.

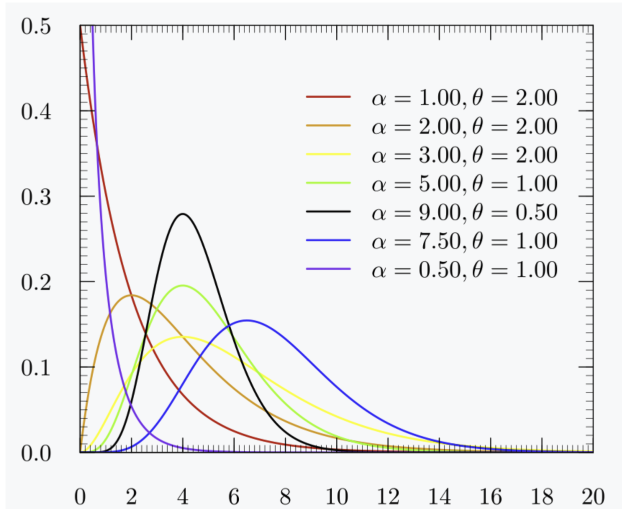

Figure 1: Gamma distributions for different values of α and λ (here $\theta = 1/\lambda$). Credit: Wikipedia.

- α — shape parameter
- λ — scale parameter (units of x, does not affect shape)

Family of probability distributions that includes special cases:

- exponential (for $\alpha = 1$)
- χ_n^2 (for $\alpha = \frac{n}{2}$, $\lambda = \frac{1}{2}$) – we'll learn more about this later
- approximates normal as $\alpha \to \infty$

Problems

1. **[4.74]** The number of offspring of an organism is a discrete random variable with mean μ and variance σ^2. Each of its offspring reproduces in the same manner. Find the expectation and variance of the number of offspring in the third generation.

   Solution:

   Let $X_1 = 1$ be the number of organisms in the first generation, X_2 be the number of offspring it has, and X_3 be the total number of their offspring (the number in the third generation). Thus we know that

   $$\mathbb{E}[X_2] = \mu, \quad \mathrm{Var}(X_2) = \sigma^2.$$

   We also have the same expectation and variance for the offspring of each second-generation organism. So

   $$\mathbb{E}[X_3 \mid X_2] = X_2\mu, \quad \mathrm{Var}(X_3 \mid X_2) = X_2\sigma^2$$

   (because $\mathrm{Var}(\sum X_i) = \sum \mathrm{Var}(X_i)$ for independent X_i's). By the law of total expectation,

   $$\mathbb{E}[X_3] = \mathbb{E}[\mathbb{E}[X_3 \mid X_2]] = \mathbb{E}[X_2\mu] = \boxed{\mu^2}.$$

By the law of total variance,

$$\begin{aligned}
\operatorname{Var}(X_3) &= \operatorname{Var}(\mathbb{E}[X_3 \mid X_2]) + \mathbb{E}[\operatorname{Var}(X_3 \mid X_2)] \\
&= \operatorname{Var}(X_2\mu) + \mathrm{E}[X_2\sigma^2] \\
&= \sigma^2\mu^2 + \mu\sigma^2 \\
&= \boxed{\sigma^2\mu(\mu+1)}.
\end{aligned}$$

2. **[5.18]** Suppose that a company ships packages that are variable in weight, with an average weight of 15 lb and a standard deviation of 10 lb. Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variables, find the probability that 100 packages will have a total weight exceeding 1700 lb.

   Solution:

   Use the CLT

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

   with

$$\mu = 15, \quad \sigma = 10.$$

   Letting Φ denote the CDF of $N(0,1)$,

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{100} X_i > 1700\right) &= \mathbb{P}(\bar{X}_{100} > 17) \\
&= \mathbb{P}\left(\frac{\bar{X}_{100} - 15}{10/\sqrt{100}} > 2\right) \\
&\approx 1 - \Phi(2) \\
&= 1 - 0.9772 = \boxed{0.0228}
\end{aligned}$$

3. Find the mean and variance of a gamma-distributed random variable

$$X \sim \operatorname{Gamma}(\alpha, \lambda).$$

   (a) Find the moment generating function of the gamma distribution.

   (b) Find the mean and variance of X.

   Solution:

   (a)

$$\begin{aligned}
M(t) = \mathbb{E}[e^{tX}] &= \int_0^\infty e^{tx}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha-1}e^{-x(\lambda-t)}\,dx
\end{aligned}$$

   We know from the pdf of the gamma distribution that

$$\int_0^\infty \frac{L^A}{\Gamma(A)}x^{A-1}e^{-Lx}\,dx = 1 \implies \int_0^\infty x^{A-1}e^{-Lx}\,dx = \frac{\Gamma(A)}{L^A}.$$

So letting $A = \alpha$, $L = \lambda - t$, we get

$$\int_0^\infty x^{\alpha-1} e^{-x(\lambda-t)}\, dx = \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}.$$

So

$$M(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha.$$

(b) To find the moments, we differentiate $M(t)$:

$$M'(t) = \alpha \left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \left(\frac{\lambda}{(\lambda-t)^2}\right) = \frac{\alpha}{\lambda}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha+1}$$

$$M''(t) = \frac{\alpha(\alpha+1)}{\lambda}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha}\left(\frac{\lambda}{(\lambda-t)^2}\right) = \frac{\alpha(\alpha+1)}{\lambda^2}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha+2}$$

So

$$M'(0) = \boxed{\mathbb{E}X = \frac{\alpha}{\lambda}}$$

$$M''(0) = \mathbb{E}[X^2] = \frac{\alpha(\alpha+1)}{\lambda^2}$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \boxed{\frac{\alpha}{\lambda^2}}.$$

4. Show that if $X_i \overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda)$, then

$$Y = \sum_{i=1}^n X_i \sim \mathrm{Gamma}(n, \lambda).$$

Solution:

Recall that $\mathrm{Exp}(\lambda) = \mathrm{Gamma}(1, \lambda)$. We already found the mgf of $\mathrm{Gamma}(\alpha, \lambda)$,

$$\left(\frac{\lambda}{\lambda-t}\right)^\alpha,$$

so the mgf of $\mathrm{Exp}(\lambda)$ is

$$\frac{\lambda}{\lambda-t}.$$

Since the X_i's are independent,

$$M_Y(t) = \mathbb{E}\left[e^{tY}\right] = \mathbb{E}\left[e^{t\sum_{i=1}^n X_i}\right] = \mathbb{E}\left[\prod_{i=1}^n e^{tX_i}\right]$$

$$= \prod_{i=1}^n \mathbb{E}\left[e^{tX_i}\right] = \prod_{i=1}^n \left(\frac{\lambda}{\lambda-t}\right) = \left(\frac{\lambda}{\lambda-t}\right)^n.$$

This is the same as the mgf of $\mathrm{Gamma}(n, \lambda)$, and since we know that the mgf uniquely describes distributions, we know that

$$Y \sim \mathrm{Gamma}(n, \lambda).$$