Mathematical Statistics Recitation 1

Section 002 (Prof. Niles-Weed)

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Disclaimer: These notes have been transcribed from handwritten notes by ChatGPT. I have done my best to proof-read and clean them up, but beware of mistakes, and email me if you find them!

Introduction/Announcements

- Office Hours: Wednesday 12–2pm, WWH 905
- Email: sonia.reilly@nyu.edu. Email me with questions or to schedule a time to meet outside of office hours.
- Format – lecture review + worked problems (will post notes + solutions)
- Recitation is for you, to help with HW + exams. No quizzes or attendance grade.
- I do not write or grade HW or exams, I only teach recitation. I can point you in the right direction on HW, but can't answer grading questions.

Lecture Review

Lecture 1

Probability Measure Definition + Properties

- nonnegative $P(A)$, sum to 1
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ disjoint probabilities $\Rightarrow$ additive

Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Law of Total Probability

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)P(B_i)$$

where B_i's are a disjoint partition of Ω.

Bayes Law

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)}$$

direct consequence of conditional probability

Independence

$$P(A \cap B) = P(A)P(B) \iff A \text{ and } B \text{ are independent}$$

Lecture 2

Cumulative Distribution Function

$$F_X(x) = P(X \leq x)$$

Probability Density Function (for continuous random variables)

$$f(x) = F'(x) \quad \text{if } F \text{ is differentiable}$$

Random Variables

- discrete: Bernoulli, binomial
- continuous: uniform, exponential, normal

Lecture 3

Functions of Random Variables

Given $Y = g(X)$ and $f_X(x)$, find $f_Y(y)$

- 1D, method 1:

  $$F_Y(y) = P(Y \leq y) = P(g(X) \leq y),$$

  solve $g(X) \leq y$ for X, then express in terms of $F_X(x)$. Finally take a derivative in y to go from $F_Y(y)$ to $f_Y(y)$.

- 1D, method 2 (equivalent given the right derivatives exist and are unique):

  $$f_Y(y) = f_X(x)\,|g'(x)|^{-1}$$

- 2D: Now $(U, V) = g(X, Y)$. Use determinant of Jacobian:

  $$f_{U,V}(u, v) = f_{X,Y}(x, y) \cdot |J(x, y)|^{-1}$$

Joint Distributions of Random Variables

(Assume continuous – all the same ideas apply to pmfs for discrete)

- marginal:

  $$f_Y(y) = \int f_{X,Y}(x, y)\,dx$$

  (no limits implies integral over whole domain)

- conditional:

  $$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

- independent:

  $$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

Lecture 4

Expectation

$$\mathbb{E}X = \sum_x x\, p_X(x) \quad \text{or} \quad \mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Both of these are just a weighted average.
If $Y = g(X)$:

$$\mathbb{E}Y = \sum_x g(x) p_X(x) \quad \text{or} \quad \mathbb{E}Y = \int g(x) f_X(x)\, dx$$

If X and Y are independent,

$$\mathbb{E}(XY) = (\mathbb{E}X)(\mathbb{E}Y)$$

Linearity of Expectation:

$$\mathbb{E}\left(\sum_{i=1}^n b_i X_i\right) = \sum_{i=1}^n b_i\, \mathbb{E}[X_i].$$

Variance

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$$

$$\sqrt{\mathrm{Var}(X)} = \text{standard deviation}$$

$$\mathrm{Var}(a + bX) = b^2 \mathrm{Var}(X)$$

If X_i are independent,

$$\mathrm{Var}\left(\sum_i X_i\right) = \sum_i \mathrm{Var}(X_i)$$

Inequalities

- Markov: If $X \geq 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

- Chebyshev:

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

Problems

1. **[1.69 and 1.70]** True or False?

   (a) If A and B are disjoint, can they be independent?

   (b) If $A \subset B$, can A and B be independent?

   Solution:

   (a) Yes, if $P(A) = 0$ or $P(B) = 0$.
   Because if A and B are disjoint, $P(A \cap B) = 0$, so for independence

$$P(A \cap B) = P(A)P(B) = 0 \implies P(A) = 0 \text{ or } P(B) = 0.$$

(b) Yes, if $P(A) = 0$ or $P(B) = 1$.
If $A \subset B$, then $P(A \cap B) = P(A)$, so if A and B are independent,

$$P(A) = P(A)P(B),$$

which implies $P(B) = 1$ or $P(A) = 0$.

2. **[2.56]** If $X \sim N(0, \sigma^2)$, find the density of $Y = |X|$.

Solution:

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) \\ &= \int_{-y}^{y} f_X(x)\,dx = \int_{-y}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}\,dx \\ &= 1 - 2F_X(-y) \end{aligned}$$

$$\Rightarrow f_Y(y) = F_Y'(y) = 2\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/(2\sigma^2)}\right), \quad y \ge 0.$$

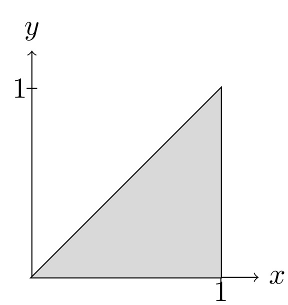
(We just double the pdf on the positive real line!)
Note: not true if $X \sim N(\mu, \sigma^2)$, $\mu \neq 0$.

3. **[3.18]** Let X and Y have the joint density function

$$f(x, y) = \begin{cases} k(x - y) & 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the region over which the density is positive.
(b) Find k. (Why can we determine k from the information given?)
(c) Find the marginal density of X.
(d) Find the conditional density of Y given X.

Solution:

(a) 

(b) We know that

$$\iint f(x, y)\,dx\,dy = 1,$$

in order for $f(x, y)$ to be the pdf of a probability distribution, so we compute

$$\begin{aligned}
\int_0^1 \int_y^1 k(x-y)\,dx\,dy &= \int_0^1 \left[k\frac{x^2}{2} - kxy\right]_y^1 dy \\
&= \int_0^1 \left[k\left(\frac{1}{2} - \frac{y^2}{2}\right) - k(1-y)y\right] dy \\
&= k\left[\frac{y}{2} - \frac{y^3}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right]_0^1 \\
&= \frac{1}{6}k = 1 \\
&\Rightarrow \boxed{k = 6}.
\end{aligned}$$

(c)

$$\begin{aligned}
f_X(x) &= \int_0^x 6(x-y)\,dy \\
&= 6xy - 3y^2\Big|_0^x \\
&= 6x^2 - 3x^2 = 3x^2.
\end{aligned}$$

(d)

$$f_{Y|X}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6(x-y)}{3x^2} = \frac{2(x-y)}{x^2}.$$

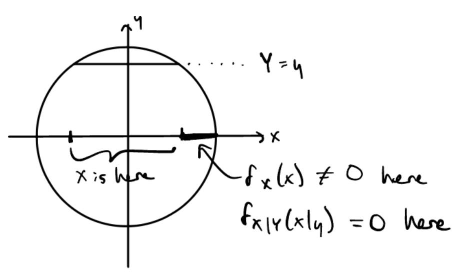
4. **[3.31]** Suppose that (X, Y) is uniform on the disk of radius 1. Without doing any calculations, argue that X and Y are not independent.

Solution:

Use another equivalent expression of independence:

$$f_{X|Y}(x \mid y) = f_X(x).$$

This cannot be true in this case. Suppose I tell you $Y = y$. Then you at least know what interval X is in. The probability $f_{X|Y}(x \mid y) = 0$ outside that interval, while $f_X(x) \neq 0$ for the same x, so X and Y are not independent.



5. **[4.3]** Let X be uniformly distributed on the interval $[1, 2]$. Find $\mathbb{E}\left[\frac{1}{X}\right]$. Is $\mathbb{E}\left[\frac{1}{X}\right] = \frac{1}{\mathbb{E}X}$?

Solution:

$$f(x) = \begin{cases} 1, & x \in [1,2], \\ 0, & \text{otherwise}, \end{cases}$$

so

$$\mathbb{E}\left[\frac{1}{X}\right] = \int_1^2 \frac{1}{x}\,dx = \log(2) - \log(1) = \log 2.$$

Does not equal $\frac{1}{\mathbb{E}X} = \frac{1}{1.5}$.

6. Suppose I offer you a game where a fair coin is tossed repeatedly. Let X be the number of tosses until the first head appears (e.g., if the first toss is a head, $X = 1$). If $X = k$, you win 2^k dollars. How much should you pay to play a round of this game?

   Solution:

   First, the pmf of X is

   $$\mathbb{P}(X = k) = \mathbb{P}(k-1 \text{ tails and } 1 \text{ head}) = 2^{-(k-1)}2^{-1} = 2^{-k}.$$

   so $X \sim \text{Geom}(\frac{1}{2})$.

   The payoff of a round is 2^X, so the expected payoff is

   $$\mathbb{E}(2^X) = \sum_{k=1}^{\infty} P(X = k)\, 2^k = \sum_{k=1}^{\infty} 2^{-k}2^k = \sum_{k=1}^{\infty} 1 = \infty.$$

   Therefore, in theory, any finite amount that you pay to play will be worth it, i.e. your expected gain will be positive.

   Sometimes expectations do not match our intuition, especially when they are infinite or do not exist (`https://en.wikipedia.org/wiki/St._Petersburg_paradox`).