Spring 2026: Mathematical Statistics
Recitation Worksheet 11
April 17, 2026

1. Which of the following are random variables, under the standard model of linear least squares regression?
   (a) x_i
   (b) y_i
   (c) e_i
   (d) σ^2
   (e) (β_0, β_1)
   (f) $(\hat{\beta}_0, \hat{\beta}_1)$
   (g) $\mathrm{Cov}(\hat{\beta}_0, \hat{\beta}_1)$
   (h) $(s^2_{\hat{\beta}_0}, s^2_{\hat{\beta}_1})$

2. The Arrhenius equation in chemistry, given by

$$k = Ae^{\frac{-E_a}{k_B T}},$$

   relates the rate of a chemical reaction k to the temperature T. For a particular reaction, we are given pairs (T_i, k_i) of rates and temperatures, and we want to find the coefficients A and E_a (k_B is a known constant). Set this up as a linear least squares problem, identifying y_i, x_i, β_0, and β_1.

3. [14.5] Three objects are located along a line at points $0 < p_1 < p_2 < p_3$. These locations are not precisely known. A surveyor makes the following measurements, each of which has some unknown amount of error.
   (a) He stands at the origin and measures the three distances from there to p_1, p_2, p_3. Let these measurements be denoted by Y_1, Y_2, Y_3.
   (b) He goes to p_1 and measures the distances from there to p_2 and p_3. Let these measurements be denoted by Y_4 and Y_5.
   (c) He goes to p_2 and measures the distance from there to p_3. Denote this measurement by Y_6.

   Using matrix notation, explain how to set up a least squares problem to find p_1, p_2, p_3 from these measurements.

4. [14.28b] Let $X_1, \ldots, X_n$ be independent random variables with means $\mu_1, \ldots, \mu_n$ and variances σ^2 for all $i = 1, \ldots, n$. Find the expectation of

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j$$

   using one of the rules we learned for random vectors. Verify your answer in a different (simpler?) way.

Selected Problems from Midterm 2

1.

(c) If $\sqrt{n}(X_n - 1) \xrightarrow{\text{d}} N(0,1)$, then $\sqrt{n}(X_n^3 - 1) \xrightarrow{\text{d}} N(0,9)$.

(d) The $(1-\alpha)$ credible interval in the Bayesian setting is also a valid $(1-\alpha)$ confidence interval in the frequentist setting.

(e) If a test always accepts the null regardless of the observed data, the corresponding p-value is 0.

3. For $X \sim N(\theta, 1)$, consider the following hypothesis testing problem:

$$H_0 : \theta = 0 \qquad \text{v.s.} \qquad H_1 : \theta = -1.$$

Recall that the pdf of $N(\theta,1)$ is $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(x-\theta)^2}{2})$.

(a) Find the rejection region of a LR test with significance level 5%. You may use $\Phi^{-1}(0.95) \approx 1.6$ and $\Phi^{-1}(0.975) \approx 2.0$, where Φ is the CDF of $N(0,1)$. *(2 points)*

(b) Find the power of your test in (a), and express your answer using Φ. *(3 points)*

4. Let $X_1, \ldots, X_n$ be i.i.d. from the Laplace distribution with density

$$f_\lambda(x) = \frac{\lambda}{2} e^{-\lambda |x|}, \qquad x \in \mathbb{R},$$

where $\lambda > 0$ is an unknown parameter. Consider the hypothesis test

$$H_0 : \lambda = \lambda_0 \qquad \text{v.s.} \qquad H_1 : \lambda \neq \lambda_0.$$

(a) Find the maximum likelihood estimator $\widehat{\lambda}$ of λ. *(2 points)*

(b) Write the generalized likelihood ratio test statistic $-2 \log LR$ in terms of $\widehat{\lambda}$, λ_0, and n. State its approximate distribution under H_0 when n is large. *(3 points)*

5. In a simple hypothesis testing problem, let $\alpha(T)$ and $\beta(T)$ denote the type-I and type-II error probabilities for a test T. For a fixed $\pi \in (0,1)$, consider the test $T^\star$ that minimizes the weighted error:

$$T^\star = \arg\min_T \, (1-\pi)\alpha(T) + \pi\beta(T).$$

(a) Show that $T^\star$ is a LR test. *(1 extra point)*

(b) Find the critical value of this LR test, in terms of π. *(1 extra point)*