Bayesian Inversion of Hierarchical PDE-Governed Inverse Problems Using INLA

Sonia Reilly

Georg Stadler

Courant Institute of Mathematical Sciences, New York University

Hierarchical PDE-Governed Bayesian Inverse Problems

- Find parameter $m(x)$ from noisy measurements y of solution u of a PDE:

$$\underbrace{y_i}_{\text{data}} = \underbrace{u[m, \boldsymbol{\theta}](x_i)}_{\text{solution}} + \underbrace{\varepsilon_i}_{\text{noise}}, \quad \text{where e.g.,} \quad -\nabla \cdot (m(x)\nabla u(x)) = f$$

- To infer discretization $\boldsymbol{m}$ of $m(x)$ with its uncertainty, compute posterior distribution using Bayes' rule

$$\underbrace{\pi(\boldsymbol{m}, \boldsymbol{\theta}|\boldsymbol{y})}_{\text{posterior}} \propto \underbrace{\pi(\boldsymbol{\theta})\pi(\boldsymbol{m}|\boldsymbol{\theta})}_{\text{priors}} \underbrace{\pi(\boldsymbol{y}|\boldsymbol{m}, \boldsymbol{\theta})}_{\text{likelihood}}$$

- Then marginalize over hyperparameters $\boldsymbol{\theta}$

$$\pi(\boldsymbol{m}|\boldsymbol{y}) = \int \pi(\boldsymbol{m}, \boldsymbol{\theta}|\boldsymbol{y})\, d\boldsymbol{\theta}$$

High-Dimensional Integrals

Working with the posterior requires integrating over high-dimensional $\boldsymbol{m}$:

- normalizing: $\int \pi(\boldsymbol{m}|\boldsymbol{y})\, d\boldsymbol{m}$
- finding mean: $\int \boldsymbol{m}\, \pi(\boldsymbol{m}|\boldsymbol{y})\, d\boldsymbol{m}$
- finding covariance: $\int \boldsymbol{m}\boldsymbol{m}^T\, \pi(\boldsymbol{m}|\boldsymbol{y})\, d\boldsymbol{m}$

Quadrature suffers from the curse of dimensionality

Common approximation methods avoid quadrature, but incur tradeoffs:

- MCMC: converges prohibitively slowly in high dimensions
- Variational inference: limited to particular families of distributions

A Tractable Case

Assumptions

- Hyperparameter $\boldsymbol{\theta}$ is low-dimensional ($\lesssim$ 10)
- Prior for $\boldsymbol{m}$ is Gaussian ("latent Gaussian field")

Latent Gaussian Model

$$\begin{aligned} \boldsymbol{\theta} &\sim \pi(\boldsymbol{\theta}) && \text{hyperparameters} \\ \boldsymbol{m} \,|\, \boldsymbol{\theta} &\sim \mathcal{N}(\boldsymbol{\mu}(\boldsymbol{\theta}), Q^{-1}(\boldsymbol{\theta})) && \text{latent Gaussian field} \\ \boldsymbol{y} \,|\, \boldsymbol{m}, \boldsymbol{\theta} &\sim \prod_i \pi(y_i \,|\, u[\boldsymbol{m}, \boldsymbol{\theta}]_i) && \text{observations} \end{aligned}$$

for some forward operator $\boldsymbol{m}, \boldsymbol{\theta} \to \boldsymbol{u}[\boldsymbol{m}, \boldsymbol{\theta}]$

INLA: Fast Approximate Marginal Posteriors for LGMs

Integrated Nested Laplace Approximations [a]

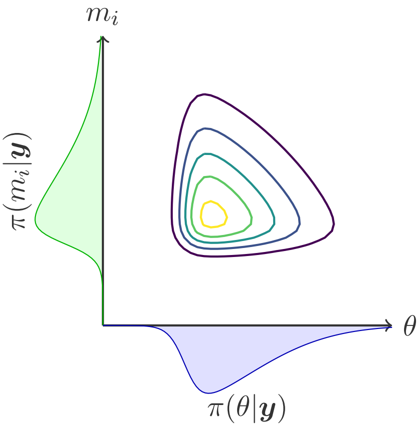
- Approximates $\pi(m_i|\boldsymbol{y})$ for LGMs

Pros

- Deterministic
- Not restricted to a family of distributions
- Fast and typically very accurate

Cons

- Does not give full joint $\pi(\boldsymbol{m}, \boldsymbol{\theta}|\boldsymbol{y})$
- **Forward model usually simple**



[a] Rue et al., *Approx. Bayesian Inference*, JRSS, 2009

INLA: A Bird's Eye View

Step 1: approximate hyperparameter distribution $\pi(\boldsymbol{\theta}|\boldsymbol{y})$

Step 2: approximate $\pi(m_i|\boldsymbol{\theta}, \boldsymbol{y})$ for all i at quadrature points $\boldsymbol{\theta}_j$

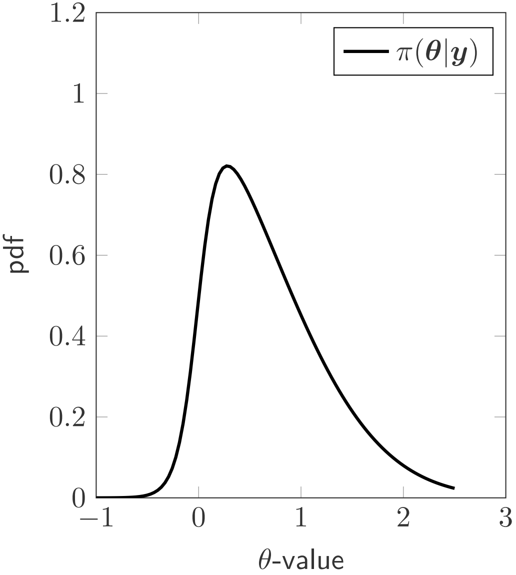
Step 3: compute $\pi(m_i|\boldsymbol{y})$ for all i using quadrature over $\boldsymbol{\theta}$:

$$\begin{aligned}\pi(m_i|\boldsymbol{y}) &= \int \pi(m_i, \boldsymbol{\theta}|\boldsymbol{y})\, d\boldsymbol{\theta} \\ &= \int \underbrace{\pi(m_i|\boldsymbol{\theta}, \boldsymbol{y})}_{\text{Step 2}}\, \underbrace{\pi(\boldsymbol{\theta}|\boldsymbol{y})}_{\text{Step 1}}\, d\boldsymbol{\theta}\end{aligned}$$

Step 1: Finding the marginal of θ

High-dimensional integral:

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) = \int \pi(\boldsymbol{m}, \boldsymbol{\theta}|\boldsymbol{y})\, d\boldsymbol{m}$$



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High-dimensional integral:

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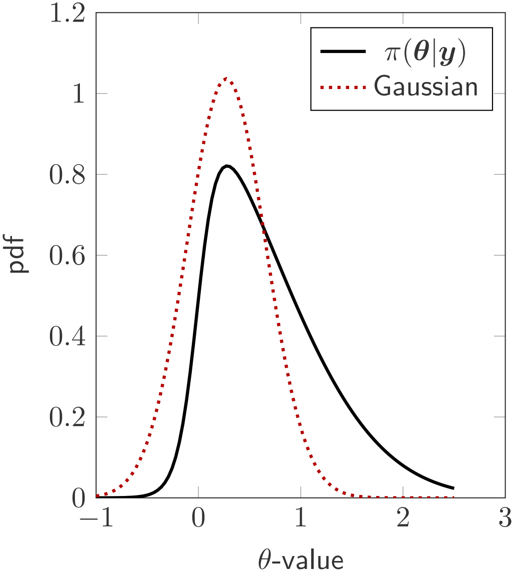
- Could use Gaussian approximation:

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) \approx \int \pi_G(\boldsymbol{m}, \boldsymbol{\theta}|\boldsymbol{y})\, d\boldsymbol{m}$$



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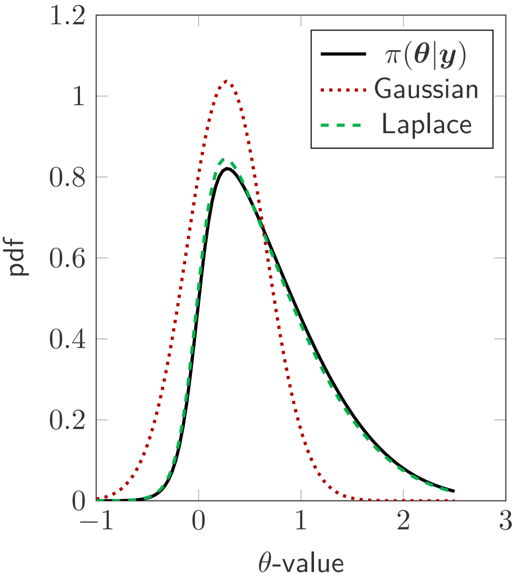
- Laplace approximation uses Gaussian approximations indirectly for much higher accuracy:

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}) \propto \frac{\pi(\boldsymbol{m}, \boldsymbol{\theta}, \boldsymbol{y})}{\pi(\boldsymbol{m}|\boldsymbol{\theta}, \boldsymbol{y})} \approx \frac{\pi(\boldsymbol{m}, \boldsymbol{\theta}, \boldsymbol{y})}{\pi_G(\boldsymbol{m}|\boldsymbol{\theta}, \boldsymbol{y})}$$

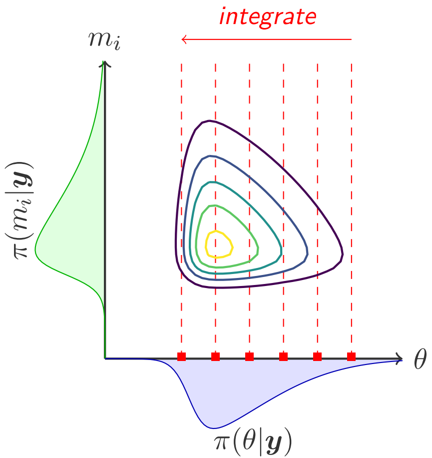


Steps 2 & 3: Finding the marginals of m_i

Low-dimensional integral:

$$\begin{aligned}\pi(m_i|\boldsymbol{y}) &= \int \pi(m_i, \boldsymbol{\theta}|\boldsymbol{y})\, d\boldsymbol{\theta} \\ &= \int \pi(m_i|\boldsymbol{\theta}, \boldsymbol{y})\, \underbrace{\pi(\boldsymbol{\theta}|\boldsymbol{y})}_{\text{Step 1}}\, d\boldsymbol{\theta}\end{aligned}$$

- Pick quadrature points in $\boldsymbol{\theta}$
- $\pi(m_i|\boldsymbol{\theta}, \boldsymbol{y}) \approx \pi_G(m_i|\boldsymbol{\theta}, \boldsymbol{y})$ or higher-order (2nd “nested" Laplace approx)
- Integrate with quadrature



Making INLA fast for PDE-governed problems

Classical INLA

- simple forward operator
- computational cost dominated by inverting large matrices
- speed achieved via sparse linear algebra

PDE-Governed Problems

- PDE operator
- computational cost dominated by solving forward and adjoint PDE
- can't build matrices explicitly

Key insights:

- For PDE operators, precision update $\Gamma_{\text{post}}^{-1} - \Gamma_{\text{prior}}^{-1}$ is low-rank, so can precompute using randomized numerical linear algebra
- PDE problem allows a multilevel approach: use coarse mesh to quickly find quadrature points, then fine mesh for accuracy in $\pi(m_i|\boldsymbol{y})$.

An example: initial condition inference in advection-diffusion

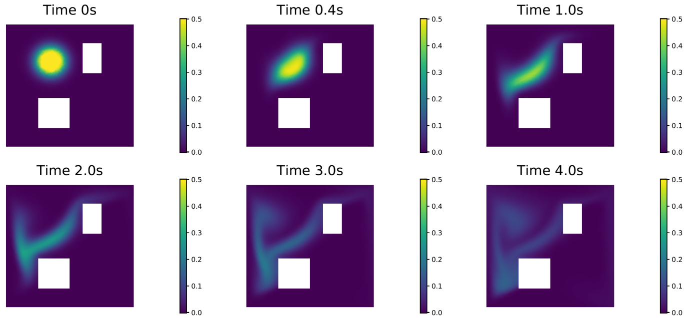
Forward problem:

- Given initial condition $u_0(x)$
- Find $u(x,t)$

$$\begin{cases} u_t - \kappa\Delta u + \boldsymbol{v}\cdot\nabla u = 0 & \text{in} \quad \Omega\times[0,T] \\ u(0,x) = u_0(x) & \text{in} \quad \Omega \\ \kappa\nabla u\cdot\boldsymbol{n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

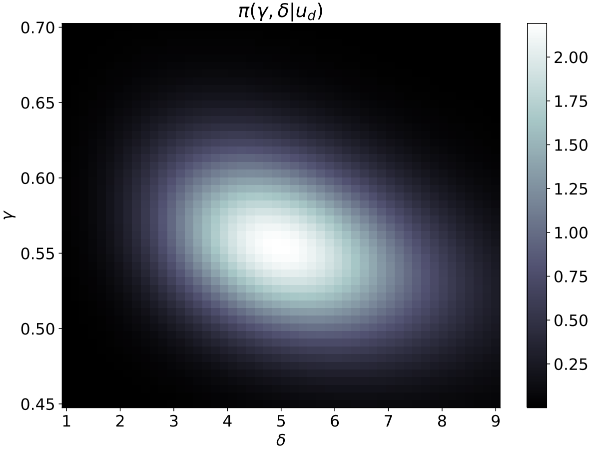
Inverse problem:

- Given data $u_d^i = u(x_i,t_i) + \varepsilon_i$ with Gaussian noise $\boldsymbol{\varepsilon}$, Gaussian prior on $\boldsymbol{u_0}$
- Find $\pi(\boldsymbol{u_0}|\boldsymbol{u_d})$



Hyperparameter Distribution

- Hyperparameters γ, δ in prior covariance $(\delta I - \gamma\Delta)^{-2}$
- γ/δ determines the length scale of the prior
- Other possible hyperparameters are magnitude of observation noise, or model parameters like diffusivity κ



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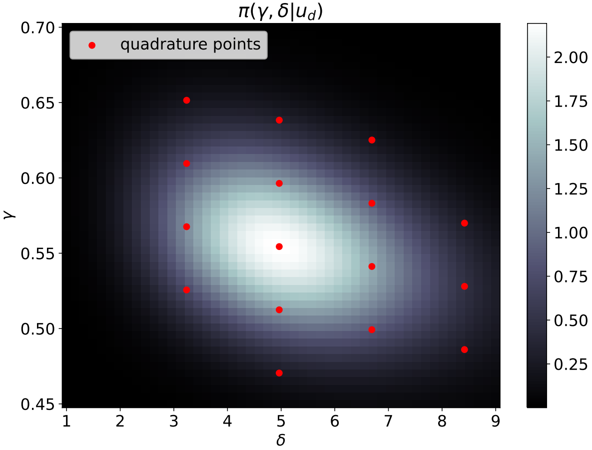
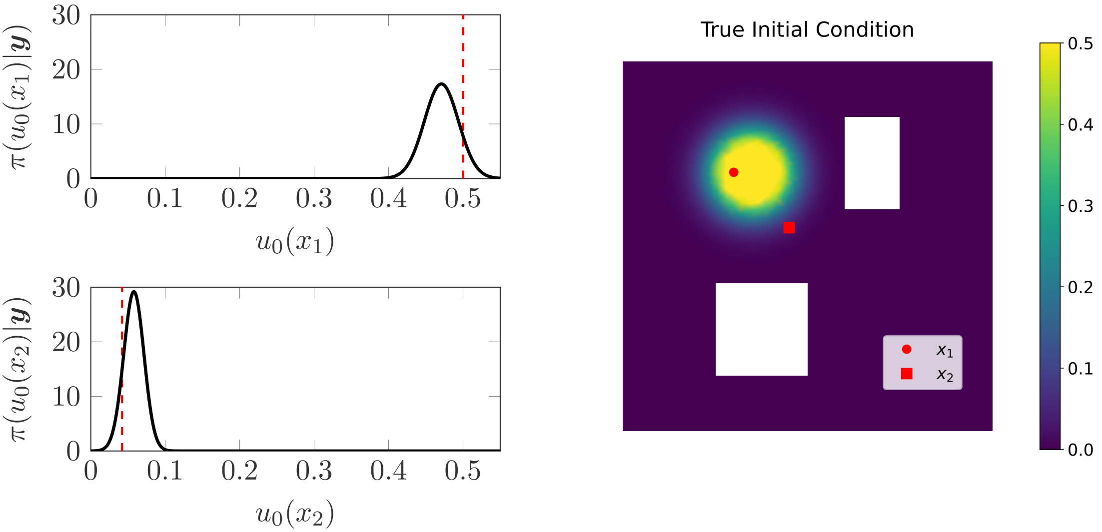


Initial Condition Distributions



Conclusion

- INLA is fast and accurate for computing pointwise uncertainty when hyperparameters are low-dimensional
- Uses Laplace approximations for high-dimensional integrals, quadrature for low-dimensional integrals
- Can be adapted to be fast for PDE-governed problems

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